Scientia Africana, Vol. 12 (No.2), December 2013. Pp23-27
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STABILITY ANALYSIS OF PREDATOR-PREY INTERACTION WITH A CROWDING EFFECT

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Received: 22-11-12
Accepted:05-12-12

ABSTRACT

The mathematical modeling of species interactions usually relies on competition models. However, it is known that species interactions may exhibit more complicated patterns with a crowding effect and this can be particularly important in benign environments. In this paper we discuss competition models with a crowding effect and indicate how they may be analyzed. We also propose a different approach to studying the global asymptotic stability of coexistence steady-state.

Keywords: Competition, Stability

INTRODUCTION

The modelling and mathematical analysis of predator-prey interaction with crowding effect is a well established research activity ([1], [2], [3], [4], [5], [7]). Experts have reported that most predator-prey interactions can be influenced by crowding in either one or both of the populations, that is, by modifying intraspecific competition or mutual interference ([1], [3], [5], [8]). One of the methods of studying these processes is by modifying the Lotka-Volterra predator and prey isoclines. In this paper, we shall include crowding factors in the growth equations for both the prey and predator. This system of equations is described by

$$(1.1) \quad \frac{dx}{dt} = x\,(A - By - Ex).$$

$$(1.2) \quad \frac{dy}{dt} = y\,(Dx - C - Fy).$$

where all constants are positive subject to the initial $x(t_0) = x_0$ and $y(t_0) = y_0$.

The parameters A and C represent the reproductive rates of prey population and predator population respectively. In this present analysis, we shall assume that the reproductive rates of growth for the two populations are positive. We assume that the crowding factors E and F are small enough relative to the other parameters that equations (1.1) and (1.2) has a solution with $x > 0$ and $y > 0$.

Although, the focus on this paper is not on studying the qualitative behaviour of solutions for this particular system of equations, it is interesting to report that the above model equations hereby specified have only three ecologically meaningful steady state solutions of which the characterizations of their stability properties can easily be studied. This paper is organized into the following sections.

Lypunov Global Asymptotic Stability

We review one of the classical notions of global asymptotic stability of steady-state which is formulated by ([2]). This result will be useful in our subsequent analysis.

Theorem (Burton, 1983)

Consider a function F which has a local minimum at the steady-state (x*, y*) of the following system of equations

(2.1) $$\frac{dx}{dt} = f(x, y)$$

(2.2) $$\frac{dy}{dt} = g(x, y)$$

Let (x*, y*) be a steady-state for the system of differential equations described by equations (1.1) and (1.2)

Let *F* be C^1 function of two variables which has a strict local minimum at (x*, y*). Consider the derivative F (x, y) = $\frac{\partial F}{\partial x}(x, y) f(x, y) + \frac{\partial F}{\partial y}(x, y) g(x, y)$ of *F* along orbits for equations (2.1) and (2.2). Let B be an open ball about (x*, y*) in the plane.

If F (x, y) = 0 for all (x, y) B, then (x*, y*) is a neutrally stable steady-state for equation 3 and 4 and the orbits of this model in B are the level sets of F.

If F (x, y) < 0 for all (x, y) ≠ (x*, y*) in B, then (x*, y*) is an asymptotically stable steady-state for this system of equations in which B lies in its basin of attraction. If F (x, y) > 0 for all (x, y) ≠ (x*, y*) in B, then (x*, y*) called unstable steady-state.

Definition: A function F which satisfies the conditions of the above theorem for system of equations (3 and 4) is called a Liapunov function for the steady-state (x*, y*).

RESULT

In this section, we shall only present the key result of this paper which concerns the notion of global stability of the positive steady-state.

Global asymptotic stability

In this section, we are interested to find out whether the coexistence steady-state is globally asymptotically stable or not. The procedure is to construct a Liapunov function F (x, y) and use the predator-prey model along with the condition of the above theorem to show that the coexistence steady-state (x*, y*) is globally asymptotically stable.

On dividing equation (2) by (1), we shall obtain

(3.1) $$\frac{dy}{dx} = \frac{y(Dx - C - Fy)}{x(A - By - Ex)}$$

By simplifying this equation, we shall obtain

(3.2) $$\frac{dy}{dx} = \frac{\left(D - \frac{C}{x} - \frac{Fy}{x}\right)}{\left(\frac{A}{y} - B - \frac{Ex}{y}\right)}$$

By separation of variables technique,

(3.3) $$\left(\frac{A}{y} - B - \frac{Ex}{y}\right) dy = \left(D - \frac{C}{x} - \frac{Fy}{x}\right) dx$$

On integrating the left hand side of this equation with respect to y and the right hand side with respect to x, we shall obtain

(3.4) $$A\ln y - By - Ex\ln y + C_1 = Dx - C\ln x - Fy\ln x$$

where C_1 is the constant of integration. From this equation,

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(3.5)
$$C_1 = Dx - C\ln x - Fy\ln x + By - A\ln y + Ex\ln y$$

We can rewrite this equation as

(3.6)
$$C_1 = D\left(x - \tfrac{C\ln x}{D} - \tfrac{Fy\ln x}{D}\right) + B\left(y - \tfrac{A\ln y}{B} + \tfrac{Ex\ln y}{B}\right)$$

Further simplification brings this equation to the form

(3.7)
$$C_1 = D\left(x - \left(\tfrac{C}{D} - \tfrac{Fy}{D}\right)\ln x\right) + B\left(y - \left(\tfrac{A}{B} + \tfrac{Ex}{B}\right)\ln y\right)$$

We have calculated that the positive steady-state or coexistence steady-state for the predator-prey model with crowding factors is (x*, y*) which can easily be verified where

(3.8) $$x^* = \frac{(AF + BC)}{(BD + EF)}$$

(3.9) $$y^* = \frac{(AD - EC)}{(BD + EF)}$$

provided AD > EC such that $C < \frac{AD}{E}$

Hence,

(3.10) $$\frac{Ex^*}{B} = \frac{AEF + BCE}{DB^2 + BEF}$$

Next, we shall use this equation to redefine a new term $\left(\frac{A}{B} - \frac{Ex^*}{B}\right)$. As such, we shall have

(3.11) $$\frac{A}{B} - \frac{Ex^*}{B} = \frac{A}{B} - \left(\frac{AEF + BCE}{DB^2 + BEF}\right)$$

By multiplying out the bracket on the right hand side of this equation, we shall obtain

(3.12)
$$\frac{A}{B} - \frac{Ex^*}{B} = \frac{ADB^2 + ABEF - ABEF - B^2CE}{B\left(DB^2 + BEF\right)}$$

By further simplification, we have obtained

(3.13) $$\frac{A}{B} - \frac{Ex^*}{B} = \frac{B^2(AD + CE)}{B^2(DB + EF)}$$

Therefore,

(3.14) $$\frac{A}{B} - \frac{Ex^*}{B} = \frac{AD - CE}{DB + EF} = y^*$$

Similarly, having calculated the value of y*

(3.15) $$\frac{Fy^*}{D} = \frac{ADF - CEF}{DB^2 + EDF}$$

Using this equation,

(3.16) $$\frac{Fy^*}{D} + \frac{Fy^*}{D} = \frac{C}{D} + \frac{ADF - CEF}{DB^2 + EDF}$$

Simplifying this equation further, it becomes

(3.17)
$$\frac{C}{D} - \frac{Fy^*}{D} = \frac{D^2(BC + AF)}{D^2(BD + EF)} = \frac{BC + AF}{BD + EF} = x^*$$

By our assumption, equation (80) becomes

(3.18)
$$C_1 = D\left(x - \left(\frac{C}{D} + \frac{Fy^*}{D}\right)\ln x\right) + B\left(y - \left(\frac{A}{B} - \frac{Ex^*}{B}\right)\ln y\right)$$

On the basis of equations (87) and (90), we shall obtain

(3.19) $$C_1 = D(x - x^*\ln x) + B(y - y^*\ln x)$$

where (x*, y*) is the positive steady-state solution of the predator-prey model with crowding factors in the positive orthant.

Hence, the Liapunov function we have been attempting to construct is

(3.20)
$$F(x,\ y) = D(x - x^* \ln x) + B(y - y^* \ln x)$$

Our next task is to use this Linpunov function to investigate the global asymptotic stability of the model under consideration.

Following similar line of analysis by Burton (1983), consider

(3.21) $F(x,\ y) - \dfrac{dF(x(t), y(t))}{dt}$

Another form of the Liapunov function which we have just constructed is

(3.22)
$$F(x(t), y(t)) = Dx(t) - Dx^* \ln x(t) + By(t) - By^* \ln y(t)$$

By implicit differentiation,

(3.23)
$$F(x,\ y) = Dx(t) - Dx^* \frac{x(t)}{x(t)} + By(t) - By^* \frac{y(t)}{y(t)}$$

Upon substituting equations (1 – 2) and simplifying, we shall obtain

(3.24) $F(x,\ y) = D(x(A - By - Ex)) - Dx^*(A - By - Ex) + B(y(Dx - C - Fy)) - By^*(Dx - C - Fy)$

This equation is further simplified to

(3.25) $F(x,\ y) = D(x - x^*)\,(A - By - Ex) + B(y - y^*)\,(D\,x - C - Fy)$

Since at the positive steady (x*, y*), we know that

(3.26) $Ex^* + By^* = A$
(3.27) $Dx^* - Fy^* = C$

Therefore,

(3.28) $F(x,\ y) = D(x - x^*)\,(Ex^* + By^* - By - Ex) + B(y - y^*)\,(Dx - Fy - Dx^* + Fy^*)$

This equation is further simplified to

(3.29) $F(x,\ y) = D(x - x^*)\,E(x^* + By^* - x) + B(y^* - y) + B(y - y^*)\,(D(x - x^*) + F(y^* - y))$

It is easy to see that x-x* = - (x*-x) and y-y* = - (y*-y). Therefore,

(3.30) $F(x,\ y) = D(x^* - x)\,E(x^* - x) + B(y^* - y)) - B(y^* - y)\,(-D(x^* - x) + F(y^* - y))$

This equation is further simplified to give

(3.31) $F(x,\ y) = ED(x^* - x)^2 - BD(x^* - x)(y^* - y)) + BD(x^* - x)(y^* - y) - BF(y^* - y)^2$

By canceling out two common terms of opposite signs, we shall obtain

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$$(3.32)\quad F(x, y) = -(ED(x^* - x)^2 + BF(y^* - y)^2)$$

Therefore,

$$(3.33)\quad F(x, y) < 0$$

for all points (x, y) ≠ (x*, y*)). Hence, for this predator-prey model with crowding factors, all orbits will approach the steady-state (x*, y*) in the positive orthant. Hence, (x*, y*) is globally asymptotically stable and the above theorem is satisfied in all dimensions.

Our key result is stated as follows:

Lemma 1. All the solutions of the model system 3 and 4 with $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$ are uniformly bounded within a region B C $\Re^2_+$ where B = (x, y) $\in C\Re^2_+$: $0 \le x \le \frac{AE + BC}{BD + EF}$, $0 \le y \le \frac{AD - CE}{BD + EF}$. The coexistence steady-state (x*, y*) is as follows: S_e = (x*, y*). The components of coexistence steady-state are given by $x^* = \frac{(AF + BC)}{(BD + EF)}$ and $y^* = \frac{(AD - CE)}{(BD + EF)}$ where the parameters are all positive constants.

Lemma 2. The coexistence steady-state is globally asymptotically stable whenever the following condition is satisfied.

*F(*x, y) < 0 for all points (x, y) ≠ (x*, y*)) where the function *F* (x, y) is called a Liapunov function for the coexistence steady-state (x*, y*).

In this paper, one has systematically shown that the positive coexistence steady-state solution of predator-prey interaction with a crowding effect is not only stable but also globally asymptotically stable under some simplifying assumptions. There is the possibility that this qualitative behaviour for a stable steady-state solution could be lost when a few important model parameters of this system of equations are varied. This question would be the subject of our next investigation.

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