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NUMERICAL BIFURCATION ANALYSIS: THE ROLE OF THE SECOND BORDER STEADY-STATE SOLUTION

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ABSTRACT

In this present deterministic numerical bifurcation analysis, if the daily intrinsic growth rate a is varied while other model parameters are fixed, the qualitative fundamental changes of the boarder steady-state solution $(0, \frac{d}{f})$ are studied. Here, the steady-state solution is stable if $a < \frac{cd}{fs}$ and $d > 0$ provided s is a positive parameter and d is one of the model parameters. It passes through a bifurcation point if $a = \frac{cd}{fs}$ and $d = 0$ and is unstable if $a > \frac{cd}{fs}$ and $d < 0$

Keywords: Bifurcation, Steady-State Solution, Stability

INTRODUCTION

The notion of a bifurcation analysis in ecological modelling is an important mathematical technique for understanding the fundamental changes in the qualitative behaviour of solutions which is due to a variation of a model parameter ([3], [4]), [5], [6]). Hence bifurcation analysis in ecological research is an active component of research. According to [1] and the several cited authors by [1] and without loss of generality, we know that, for a system of nonlinear first order differential equations, a steady-state solution can either be called stable if the signs of the eigenvalues are both negative and unstable if the signs of the eigenvalues are of opposite signs. But the bifurcation values where a stable-steady state solution changes to an unstable steady-state solution remains to be an open problem in the context of this interdisciplinary research. These bifurcation values can provide some insights to ecologists.

In the sequel, we will present a few key results of our numerical bifurcation analysis which we have not seen elsewhere.

This paper is organized into the following sections: Sections 2 and 3 will tackle the notions of bifurcation and some simplifying modelling assumptions. Sections 4 and 5 will tackle the mathematical formulation and the characterization components of this proposed problem. Section 6 is the core component of this study which tackles the problem of constructing the bifurcation points and the fundamental changes in the qualitative behaviour of the steady-state solutions. The key results which one has achieved in this paper are quantitatively discussed in section 7 and summarized in section 8.

Bifurcation Analysis

If a model parameter is varied while other parameters are fixed, we can study the fundamental changes in the qualitative behaviour of steady-state solutions and hence find the bifurcation points where a stable steady-state solution changes to an unstable steady-state solution. For example, after linearizing the interaction continuous and partially differentiable functions in the neighbourhood of an arbitrary steady-state solution, we will aim to characterize the stability and instability behaviour of the steady-state solution qualitatively. In this respect, we can explore the standard mathematical technique of the changes in the sings of the eigenvalues to specify if a steady-states solution is either stable or unstable. In some instances, a steady-state solution can be characterized as sitting on the cusp. A systematic calculation where a steady-state changes from a stable node to a saddle can have interesting application in the study of biological interaction which is both attractive and cost-effective.

This numerical bifurcation analysis can be useful in ecological monitoring and stability. For other sophisticated bifurcation methods, see [3].

Modelling Assumptions

In this paper, our core assumptions will border on the linear Malthusian growth phenomenon, logistic population growth and the law of mass action which are central in the formulation of a system of first order differential equations that describe the interspecific competition between two plant species in a Lotka-Voltera sense ([1]. [2]).

Mathematical Formulation

Recently, [2] introduced a mathematical model of plant species interaction in a harsh climate. They consider whether interactions between the species change in character as environment change. The model is constructed based on the notion of a summer season when the plants grow, followed by a winter season when there is no growth but when the plants are subject to the effects of events such as winter storms, see also [1].

The model of competition has the following form

$$(4.1) \qquad \frac{dy}{dt} = \alpha_1 y(t)(1-\beta_1 y(t)-\gamma_1 z(t))$$

$$(4.2) \qquad \frac{dz}{dt} = \alpha_2 z(t)(1-\beta_2 z(t)-\gamma_2 y(t))$$

Here y and z denote the population of two plant species at time *t*. Here the non-negative constants α_i, β_i, γ_i, i = 1,2 are given respectively, as the intrinsic growth rate, the intra-species competitive parameter and the inter-species competitive parameter. This model equation has four steady-states.

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$y = 0, \quad z = 0$

$y = 0, \ z = \frac{1}{\beta_2},$

$y = \frac{1}{\beta_1}, \ z = 0$

$y = \frac{\beta_2 - \gamma_1}{\beta_1\beta_2 - \gamma_1\gamma_2}, \quad z = \frac{\beta_1 - \lambda_2}{\beta_1\beta_2 - \gamma_1\gamma_2}$

They discussed how to choose the parameter value α_i, β_i, γ_i, i = 1,2 such that the model is reasonable. They noticed that although the variation in α_i, β_i, γ_i, i = 1,2 between the species is quite small, the behaviour of two such close species are much different over a growing season of several years length. The population of one species may die away and would become extinct over a growing season of several years length. They pointed that small perturbation in the environment could have quite devastating and unexpected results for ecosystems. Some steady-states are stable. For the purpose of this paper, we will consider a simplified version of the above model equations ([1]) such as

(4.3) $$\frac{dN_1}{dt} = N_1(a - bN_1 - cN_2),$$

(4.4) $$\frac{dN_2}{dt} = N_2(d - eN_1 - fN_2),$$

where the initial conditions are $N_1(0) = N_{10} > 0$ and $N_2(0) = N_{20} > 0$.

For the purpose of this bifurcation analysis, we will consider the following system of first order nonlinear ordinary differential equations

(4.5) $$\frac{dN_1}{dt} = N_1(sa - bN_1 - cN_2),$$

(4.6) $$\frac{dN_2}{dt} = N_2(d - eN_1 - fN_2),$$

Similarly the initial conditions are $N_1(0) = N_{10} > 0$, $N_2(0) = N_{20} > 0$ and $s > 0$.

Characterization of Steady-State Solutions

If the rates of change are equated to zero and the interactions functions are solved analytically, we will obtain the four steady-state solutions namely (0,0), $\left(\frac{sa}{b}, 0\right)$, $\left(0, \frac{d}{f}\right)$, and (N_{1e}, N_{2e}) where $N_{1e} = \frac{asf - cd}{bf - ce}$ and $N_{2e} = \frac{bd - eas}{bf - ce}$ provided $sa > \frac{cd}{sf}$, $sa < \frac{bd}{e}$, $bf > ce$.

By using a standard mathematical technique of linearization at an arbitrary steady-state solution (N_{1e}, N_{2e}), we will consider two interaction functions $F(N_{1e}, N_{2e})$ and $G(N_{1e}, N_{2e})$ which are assumed to be partially differentiable and continuous at this arbitrary steady-state solution (N_{1e}; N_{2e}).

In our context, the mathematical structures of these two functions are

(5.1) $$F(N_{1e}, N_{2e}) = asN_{1e} - bN_{1e}^2 - cN_{1e}N_{2e},$$

(5.2) $$G(N_{1e}, N_{2e}) = dN_{2e} - eN_{1e}N_{2e} - fN_{2e}^2$$

To determine the stability property of each steady-state solution, we differentiated these two functions partially with respect to N_{1e} and N_{2e} and obtain the following Jacobian coefficients such as

(5.3) $J_{11} = \dfrac{\partial F}{dN_{1e}} = as - 2bN_{1e} - cN_{2e},$

(5.4) $J_{12} = \dfrac{\partial F}{dN_{2e}} = -cN_{1e},$

(5.5) $J_{21} = \dfrac{\partial G}{dN_{1e}} = -eN_{2e},$

(5.6) $J_{22} = \dfrac{\partial G}{dN_{2e}} = d - eN_{1e} - 2fN_{2e}$

Upon evaluating these values of partial derivatives at each steady-state solution, we can set up a Jacobian matrix from which two eigenvalues can be calculated. For example, at the steady-state $(0, \frac{d}{f})$, the two eigenvalues which are unique to these model parameters are if $\lambda_1 = as - \frac{cd}{f}$ *and* $\lambda_2 = -d$

Numerical Bifurcation Analysis

(1) If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $a < \frac{cd}{fc}$ and $d > 0$. This first observation indicates that the steady-state solution $(0, \frac{d}{f})$ is stable.

(2) If $\lambda_1 < 0$ and $\lambda_2 = 0$, then $a < \frac{cd}{fs}$ and $d = 0$ This second observation indicates that the steady-state solution $(0, \frac{d}{f})$ is sitting on the cusp

(3) If $\lambda_1 = 0$ and $\lambda_2 < 0$, then $a < \frac{cd}{fs}$ and $d > 0$. This third observation indicates that the steady-state solution $(0, \frac{d}{f})$ is sitting on the cusp

(4) If $\lambda_1 = 0$ and $\lambda_2 = 0$, then $a = \frac{cd}{fs}$ and $d = 0$. The fourth observation indicates that the loss of the stability for the steady-state solution $(0, \frac{d}{f})$.

(5) If $\lambda_1 = 0$ and $\lambda_2 > 0$, then $a = \frac{cd}{fs}$ and $d < 0$. The fifth observation indicates that the steady-state solution $(0, \frac{d}{f})$ is sitting on the cusp.

(6) If $\lambda_1 > 0$ and $\lambda_2 = 0$, then $a > \frac{cd}{fs}$ and $d = 0$. This sixth observation indicates that the steady-state solution $(0, \frac{d}{f})$ is sitting on the cusp.

(7) If $\lambda_1 < 0$ and $\lambda_2 > 0$, then $a < \frac{cd}{fs}$ and $d > 0$. The Seventh observation indicates that the steady-state solution $(0, \frac{d}{f})$ is unstable. Here, the instability criteria of the steady-state solution $(0, \frac{d}{f})$ are the partial opposite of the stability criteria of the steady-state solution $(0, \frac{d}{f})$.

(8) If $\lambda_1 > 0$ and $\lambda_2 < 0$, then $a > \frac{cd}{fs}$ and $d > 0$. This eighth observation indicates that the steady-state solution $(0, \frac{d}{f})$ is unstable. Similarly, the instability criteria of the steady-state solution $(0, \frac{d}{f})$.

(9) If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $a > \frac{cd}{fs}$ and $d < 0$. This nineth observation indicates that the steady-state solution $(0, \frac{d}{f})$ is unstable. We can see clearly that the instability criteria of the steady-state solution $(0, \frac{d}{f})$ are the complete opposite of the stability criteria of the steady-state solution ($0, \frac{d}{f}$).

DISCUSSION

In this present analysis, the steady-state solution $(0, \frac{d}{f})$ is a stable node for $a <$ $(0, \frac{cd}{fs})$ and $d > 0$. It is a saddle for $a <> \frac{cd}{fs}$ and $d < 0$.

Therefore, the steady-state solution will change from a stable node to a saddle as it persists through tire bifurcation point $a = \frac{cd}{fs}$ and $d = 0$.

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In this study, we have found that the steady-state solution $(\frac{sa}{b}, 0)$ is stable when the growth rate of the first population is strictly less than a multiple of the growth rate of the second population and when the growth rate of the second population is strictly greater than zero.

This steady-state solution is similarly unstable when the growth rate of the first population is strictly greater than a multiple of the growth rate of the second population and when the growth rate of the second population is strictly less than zero.

A bifurcation point $a = \frac{cd}{fs}$ and d = 0 has been determined which is capable of providing further insights in biosciences research.

REFERENCES

Ekaka-a, E.N., Computational and Mathematical Modelling of Plant Species Interactions in a Harsh Climate, PhD Thesis, Department of Mathematics.The University of Liverpool and the University of Chester, United Kingdom, 2009.

Ford, N.J., Lumb, P.M. and Ekaka-a, E. N, Mathematical Modelling of plant species interactions in a harsh climate, *Journal of Computational and Applied Mathematics* 234, (2010), 2732-2744.

Ford, N.J., and Norton, S.J., noise-induced changes to the behaviour of semi-implicit Euler methods for stochastic delay differential equations undergoing bifurcation, *Journal of Computational and Applied Mathematics* 229, (2009), 462 – 470

Khamis, S.A., Tchuenche J.M, Lukka, M. and Heilio M. (2011). Dynamics of fisheries with prey reserve and harvesting. *International Journal of Computer Mathematics* Vol. 88 (No 8), pp 1776 - 1802

Troost, T.A., Kooi, B.W., and Kooijman, S.A.L.M., Bifurcation analysis of ecological and evolutionary processes in ecosystems, Ecological Modelling 204, (2007), 253-268.

Uka, U.A. and Ekaka-a, E.N. (2012).Numerical Simulation of Interacting Fish Populations with Bifurcation.*Scientia Africana*, Vol. II (No 1), June 2012, pp 121 - 124